J. Fluid Mech. (1965), *vol.* 21, *part* 4, *pp.* 577–590
Printed in Great Britain

Magnetohydrodynamic flow in rectangular ducts

By J. C. R. HUNT
Central Electricity Research Laboratories, Leatherhead*

(Received 27 July 1964)

The paper presents an analysis of laminar motion of a conducting liquid in a rectangular duct under a uniform transverse magnetic field. The effects of the duct having conducting walls are investigated. Exact solutions are obtained for two cases, (i) perfectly conducting walls perpendicular to the field and thin walls of arbitrary conductivity parallel to the field, and (ii) non-conducting walls parallel to the field and thin walls of arbitrary conductivity perpendicular to the field.

The boundary layers on the walls parallel to the field are studied in case (i) and it is found that at high Hartmann number (M), large positive and negative velocities of order MV_c are induced, where V_c is the velocity of the core. It is suggested that contrary to previous assumptions the magnetic field may in some cases have a destabilizing effect on flow in ducts.

1. Introduction

The design of magnetohydrodynamic generators, pumps and accelerators requires an understanding of the flows of conducting fluids in rectangular ducts with finitely conducting walls under transverse magnetic fields. At the present time even the case of uniformly conducting incompressible laminar flow with no variation in the flow direction has not been fully analyzed. In this paper we confine ourselves to problems of this type alone. The main characteristics of such flows that need to be known are:

(1) the volumetric flow rate q through the duct for given pressure gradient and magnetic field;

(2) the potential difference between electrodes placed in the walls;

(3) the stability of the flow.

Three exact solutions have been found for incompressible laminar flows in ducts with transverse magnetic fields:

(1) rectangular ducts with non-conducting walls and the field perpendicular to one side (Shercliff 1953);

(2) rectangular ducts with perfectly conducting walls (Chang & Lundgren 1961; Uflyand 1961);

(3) circular pipes with non-conducting walls (e.g. Gold 1962 and Fabri & Siestrunck 1960).

Approximate methods have been developed for the physically interesting

* Seconded to the Department of Engineering Science, University of Warwick.

case of flows at high Hartmann number, M. For a rectangular duct with non-conducting walls Shercliff (1953) developed an approximate method for analysing the boundary layers on the walls parallel to the field and thence deduced q and the potential distributions round the walls. By ignoring the reduction in flow rate due to the boundary layers, he then found a first approximation for q in a duct of any cross-section, which was later extended to the case of ducts with thin walls of any conductivity by Chang & Lundgren (1961). Sakao (1962) used a variational method to find a second-order approximation for q in circular pipes.

In finding the overall features of the flow at high M, approximate methods are often best since the exact solutions are in the form of infinite series whose rate of convergence *decreases* for higher values of M. It is possible, however, to compare the expressions for q at high M obtained by the two methods. The only case hitherto of the approximate expression for q at high M agreeing with that derived from the exact solution is that of flow in a rectangular duct with non-conducting walls (Williams 1963). In the same paper, using some lengthy mathematics, Williams deduced an expression for q at high M from Chang & Lundgren's result for flow in a rectangular channel with perfectly conducting walls. The asymptotic form of the exact solution for circular pipe flow at high M provides an expression for q which differs from Shercliff's (1962*a*) and Sakao's (1962) approximate expressions by a term due to the velocity defect in the boundary layers.

No satisfactory approximate or exact solutions exist for the most important practical case of a rectangular duct with conducting walls parallel to the field and non-conducting walls perpendicular to the field. Some observations on this problem have been made by Shercliff (1962*b*, page 16) and by Braginskii (1960). Grinberg (1961, 1962) has attempted an exact analysis using a Green's function method but his result is incomplete. We have not been able to solve this problem, but we have solved exactly two other classes of problem of flow in a rectangular duct, in which the duct has (i) perfectly conducting walls at right angles to the field and thin walls of arbitrary conductivity parallel to the field, and (ii) non-conducting walls parallel to the field and thin walls of arbitrary conductivity perpendicular to the field. These exact solutions are in the form of infinite series whose rate of convergence *increases* for higher values of M. In this paper we examine the asymptotic form of these solutions at high M and draw some interesting physical conclusions, the main one being that a magnetic field may have a destabilizing influence on the flow in a duct.

2. Formulation of the problem

We consider the steady flow of an incompressible conducting fluid driven by a pressure gradient along a rectangular duct under an imposed transverse magnetic field. We assume that the walls of the duct are thin and finitely conducting. We postulate no secondary flow and no variation in duct cross-section or magnetic field. Consequently all conditions except pressure are constant along the duct.

Relative to the axes defined in figure 1, the equations describing such magnetohydrodynamic duct flows are:

$$j_x = \sigma(-\partial\phi/\partial x - v_z B_0), \quad j_y = \sigma(-\partial\phi/\partial y), \tag{1}$$

$$\partial j_x/\partial x + \partial j_y/\partial y = 0, \tag{2}$$

$$j_x = \partial H_z/\partial y, \quad j_y = -\partial H_z/\partial x, \tag{3}$$

and

$$0 = -\frac{\partial p}{\partial z} + j_x B_0 + \tilde{\eta}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v_z. \tag{4}$$

j_x and j_y are the current components; ϕ is the electric potential; H_z is the induced field which may also be considered as a current stream function; B_0 is the flux density of the imposed magnetic field; v_z is the velocity; $\sigma, \tilde{\eta}, \partial p/\partial z$ are conductivity, viscosity and pressure gradient respectively. Let $2a$, and $2b$ be the lengths of the sides of the channel (see figure 1).

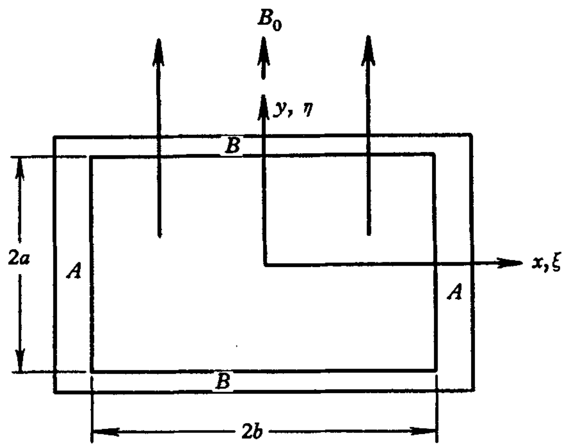

FIGURE 1. Cross-section of a rectangular duct with magnetic field in y-direction. The walls AA lie at $x = \pm b$ and BB at $y = \pm a$.

The equations are usually re-written to give two coupled second-order partial differential equations in H_z and v_z,

$$0 = -\frac{\partial p}{\partial z} + B_0 \frac{\partial H_z}{\partial y} + \tilde{\eta}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v_z \tag{5}$$

and

$$0 = B_0 \frac{\partial v_z}{\partial x} + \frac{1}{\sigma}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) H_z. \tag{6}$$

If v_z and H_z are normalized in terms of $(\partial p/\partial z)$, a, σ and $\tilde{\eta}$, (5) and (6) become

$$\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} + M\frac{\partial H}{\partial \eta} = -1, \tag{7}$$

and

$$\frac{\partial^2 H}{\partial \xi^2} + \frac{\partial^2 H}{\partial \eta^2} + M\frac{\partial V}{\partial \eta} = 0, \tag{8}$$

where

$$V = v_z\tilde{\eta}\bigg/\left(-\frac{\partial p}{\partial z}\right)a^2; \quad H = H_z(\tilde{\eta})^{\frac{1}{2}}\bigg/\left(-\frac{\partial p}{\partial z}\right)a^2\sigma^{\frac{1}{2}};$$

$$M = aB_0(\sigma/\tilde{\eta})^{\frac{1}{2}}; \quad \xi = x/a; \ \eta = y/a.$$

Also let $$\Phi = \phi\,(\sigma\tilde{\eta})^{\frac{1}{2}} \Big/ \left(-\frac{\partial p}{\partial z}\right) a^2 \quad \text{and} \quad l = b/a.$$

We follow Shercliff (1956) in the specification of boundary conditions on H at a thin bounding wall. If $\partial H/\partial n$ is the normalized inward normal gradient of H at the wall, the condition satisfied by H at the wall is

$$\partial H/\partial n = \sigma a H/\sigma_w w,$$

where σ_w is the conductivity of the wall and w its thickness ($w \ll a$). If $d = \sigma_w w/\sigma a$, the boundary condition for H is

$$\partial H/\partial n = H/d. \tag{9}$$

The boundary condition on V, of course, is that it should vanish at the walls.

In §§3 and 4 we consider the following two cases, defined with reference to figure 1:

case I rectangular duct with walls BB perfectly conducting and walls AA of arbitrary conductivity;

case II rectangular duct with walls AA non-conducting and walls BB of arbitrary conductivity.

3. Case I

In this case walls BB are perfectly conducting, $d_B = \infty$, and walls AA have arbitrary conductivity d_A. The boundary conditions on V and H are:

$$\left.\begin{aligned} &\text{at} \quad \eta = \pm 1, \quad V = 0 \quad \text{and} \quad \partial H/\partial \eta = 0, \\ \text{and} \quad &\text{at} \quad \xi = \pm l, \quad V = 0 \quad \text{and} \quad \partial H/\partial \xi = \mp H/d_A. \end{aligned}\right\} \tag{10}$$

We can satisfy the boundary conditions on $\eta = \pm 1$ by expressing V and H as Fourier series in η, with the coefficients functions of ξ,

$$V = \sum_{j=0}^{\infty} v_j(\xi) \cos \alpha_j \eta; \quad H = \sum_{j=0}^{\infty} h_j(\xi) \sin \alpha_j \eta; \quad 1 = \sum_{j=0}^{\infty} a_j \cos \alpha_j \eta,$$

where $\alpha_j = (j+\frac{1}{2})\pi$ and $a_j = 2(-1)^j/\alpha_j$. Substituting these expansions for V and H into (7) and (8) leads to two ordinary differential equations for v_j and h_j,

$$v_j'' - \alpha_j^2 v_j + M\alpha_j h_j = -a_j,$$

and

$$h_j'' - a_j^2 h_j - M\alpha_j v_j = 0.$$

The solutions of these equations which satisfy the boundary conditions on $\xi = \pm l$ are

$$v_j = \frac{a_j}{\alpha_j^2 + M^2}\left[1 - \frac{([1 - iM/\alpha_j]\cosh r_{2j} l + d_A r_{2j} \sinh r_{2j} l)\cosh r_{1j}\xi}{K_j} - \frac{([1 + iM/\alpha_j]\cosh r_{1j} l + d_A r_{1j} \sinh r_{1j} l)\cosh r_{2j}\xi}{K_j}\right], \tag{11}$$

$$h_j = \frac{a_j}{\alpha_j^2 + M^2}\left[-\left(\frac{M}{\alpha_j}\right) + \frac{([i + M/\alpha_j]\cosh r_{2j} l + i d_A r_{2j} \sinh r_{2j} l)\cosh r_{1j}\xi}{K_j} - \frac{([i - M/\alpha_j]\cosh r_{1j} l + i d_A r_{1j} \sinh r_{1j} l)\cosh r_{2j}\xi}{K_j}\right], \tag{12}$$

where $r_{1j}, r_{2j} = (\alpha_j^2 \pm iM\alpha_j)^{\frac{1}{2}}$

and

$$K_j = 2\cosh r_{1j} l \cosh r_{2j} l + d_A[r_{2j}\sinh r_{2j} l \cosh r_{1j} l + r_{ij}\sinh r_{ij} l \cosh r_{2j} l].$$

Now r_{1j} and r_{2j} may be split into their real and imaginary parts, namely,

$$r_{1j}, r_{2j} = \beta_j \pm i\gamma_j,$$

where $\beta_j, \gamma_j = (\tfrac{1}{2}\alpha_j)^{\frac{1}{2}}[\pm\alpha_j + (\alpha_j^2 + M^2)^{\frac{1}{2}}]^{\frac{1}{2}}.$

Hence $K_j = \cosh 2\beta_j l + \cos 2\gamma_j l + d_A(\beta_j \sinh 2\beta_j l - \gamma_j \sin 2\gamma_j l).$

The final result for V and H is

$$V = \sum_{j=0}^{\infty} \frac{2(-1)^j \cos\alpha_j\eta}{\alpha_j(\alpha_j^2+M^2)}\left[1 - \frac{C_j(\xi) - M/\alpha_j D_j(\xi)}{K_j} - \frac{d_A[\beta_j E_j(\xi) - \gamma_j F_j(\xi)]}{K_j}\right], \quad (13)$$

$$B = \sum_{j=0}^{\infty} \frac{2(-1)^j \sin\alpha_j\eta}{\alpha_j(\alpha_j^2+M^2)}\left[-\left(\frac{M}{\alpha_j}\right) + \frac{M/\alpha_j C_j(\xi) + D_j(\xi)}{K_j} + \frac{d_A[\gamma_j E_j(\xi) + \beta_j F_j(\xi)]}{K_j}\right], \quad (14)$$

where

$$C_j(\xi) = \cos\gamma_j(l-\xi)\cosh\beta_j(l+\xi) + \cos\gamma_j(l+\xi)\cosh\beta_j(l-\xi),$$
$$D_j(\xi) = \sin\gamma_j(l-\xi)\sinh\beta_j(l+\xi) + \sin\gamma_j(l+\xi)\sinh\beta_j(l-\xi),$$
$$E_j(\xi) = \cos\gamma_j(l-\xi)\sinh\beta_j(l+\xi) + \cos\gamma_j(l+\xi)\sinh\beta_j(l-\xi),$$
$$F_j(\xi) = \sin\gamma_j(l-\xi)\cosh\beta_j(l+\xi) + \sin\gamma_j(l+\xi)\cosh\beta_j(l-\xi).$$

If the non-dimensional volumetric flow rate, Q, is defined by

$$Q = \int_{-1}^{+1}\int_{-l}^{+l} V\,d\eta\,d\xi,$$

then, from (17),

$$Q = \sum_{j=0}^{\infty} \frac{8}{\alpha_j^2(\alpha_j^2+M^2)}\left[l - \frac{(\beta_j - M\gamma_j/\alpha_j)\sinh 2\beta_j l + (\gamma_j + M\beta_j/\alpha_j)\sin 2\gamma_j l}{(\beta_j^2+\gamma_j^2)K_j} - \frac{\alpha_j d_A(\cosh 2\beta_j l - \cos 2\beta_j l)}{(\alpha_j^2+M^2)^{\frac{1}{2}}K_j}\right]. \quad (15)$$

Note that the terms independent of ξ in the expressions for V and H are the Fourier expansions of the Hartmann solution and also that, as $d_A \to \infty$, the solutions become identical to those obtained for rectangular ducts with perfectly conducting walls by Uflyand (1961) and Chang & Lundgren (1961).

At high Hartmann numbers the fluid tends to move at a constant velocity, the core velocity, in the centre of the duct with the velocity gradients confined to narrow Hartmann layers on the walls BB. If the non-dimensional core velocity is V_c, then $V_c \sim 1/M^2$ as $M \to \infty$. The current density is also constant in the core. The Hartmann layers are well understood, but the boundary layers on the walls AA are less well understood and need examination.

We consider the boundary layer on the wall $\xi = -l$ at high Hartmann number and make the following approximations:

as $M \to \infty$, $\beta_j, \gamma_j \sim (\tfrac{1}{2}\alpha_j M)^{\frac{1}{2}}(1 \pm O(1/M)\ldots) \sim \lambda_j$, where $\lambda_j = (\tfrac{1}{2}\alpha_j M)^{\frac{1}{2}}$,

and hence $K_j \sim \tfrac{1}{2}[\exp\{2(\tfrac{1}{2}\alpha_j M)^{\frac{1}{2}} l\}](1 + d_A(\tfrac{1}{2}\alpha_j M)^{\frac{1}{2}})(1 + O(1/M))$

$$\sim \exp\tfrac{1}{2}[(2\lambda_j l)(1 + d_A\lambda_j)].$$

For clarity we take the two cases of $d_A = 0$ and $d_A = \infty$ and consider the V, H, and Φ profiles in the boundary layer. If $\xi' = \xi + l$ and $\Phi = 0$ at $\eta = \pm 1$, then for $d_A = 0$, as $M \to \infty$,

$$V \sim \sum_{j=0}^{\infty} \frac{2(-1)^j \cos \alpha_j \eta}{M^2 \alpha_j} [1 - \exp(-\lambda_j \xi')\{\cos(\lambda_j \xi') - M/\alpha_j \sin(\lambda_j \xi')\}], \qquad (16)$$

$$H \sim \sum_{j=0}^{\infty} \frac{2(-1)^j \sin \alpha_j \eta}{M^2 \alpha_j} [-(M/\alpha_j) + \exp(-\lambda_j \xi')\{(M/\alpha_j)\cos(\lambda_j \xi') + \sin(\lambda_j \xi')\}], \qquad (17)$$

$$\Phi \sim \sum_{j=0}^{\infty} \frac{2(-1)^j \cos \alpha_j \eta}{M^2 \alpha_j^2} [\{M^{\frac{3}{2}}/(2\alpha_j)^{\frac{1}{2}}\} \exp(-\lambda_j \xi')\{\cos(\lambda_j \xi') + \sin(\lambda_j \xi')\}], \qquad (18)$$

and for $d_A = \infty$, as $M \to \infty$,

$$V \sim \sum_{j=0}^{\infty} \frac{2(-1)^j \cos \alpha_j \eta}{M^2 \alpha_j} [1 - \exp(-\lambda_j \xi')\{\cos(\lambda_j \xi') - \sin(\lambda_j \xi')\}], \qquad (19)$$

$$H \sim \sum_{j=0}^{\infty} \frac{2(-1)^j \sin \alpha_j \eta}{M^2 \alpha_j} [-(M/\alpha_j) + \exp(-\lambda_j \xi')\{\cos(\lambda_j \xi') + \sin(\lambda_j \xi')\}], \qquad (20)$$

$$\Phi = \sum_{j=0}^{\infty} \frac{2(-1)^j \cos \alpha_j \eta}{\alpha_j^2 M^2} [2\lambda_j \exp(-\lambda_j \xi') \sin(\lambda_j \xi')]. \qquad (21)$$

In contrast with the exact solutions for rectangular and circular pipes with non-conducting walls, the higher terms in these series decrease exponentially and therefore it is a good approximation to consider the first few terms only. Hence we see that the V, H, and Φ boundary layer profiles approximately have the form of exponentially damped sine waves, the thickness of the layers being $O(M^{-\frac{1}{2}})$. In figures 2 and 3 velocity profiles when $d_A = 0$ are plotted for various values of η at $M = 100$ and for various values of M at $\eta = 0$. In figure 4, the velocity profiles when $d_a = \infty$ are plotted for various values of η at *arbitrary* M, provided $M \gg 1$. Here the abscissa $M^{\frac{1}{2}}\xi'$ provides a universal plot, when $M \gg 1$. Note that in all cases $V/V_c \to 1$ as $\xi' \to \infty$.

The dramatic effect on the flow of varying the conductivity of the walls AA is seen by comparing figures 3 and 4. When $d_A = \infty$ (figure 4) the maximum velocity in the boundary layer A is greater than the core velocity though of the same order; but when $d_A = 0$ (figure 3), the maximum velocity is $O(M) V_c$ and the minimum velocity is *negative* provided M is high enough. We can deduce from (16) that the maximum velocity tends to $0{\cdot}25 MV_c$ as $M \to \infty$, while the minimum velocity becomes locally negative for $M > 89$ and tends to $-0{\cdot}011 MV_c$ as $M \to \infty$. The physical reason for the effect on the flow of varying d_A when $M \gg 1$ may be seen from equation (13) which shows that the form of the velocity profile depends on $d_A M^{\frac{1}{2}}$, the ratio of the conductance of the wall to that of the boundary layer on the wall A. Thus when $d_A M^{\frac{1}{2}} \gg 1$ the currents return to the walls BB through the walls AA and when $d_A M^{\frac{1}{2}} \ll 1$ the currents return to the walls BB through the boundary layers on walls AA. These effects are shown in figures 5a and b. In the first case the $\mathbf{j} \times \mathbf{B}$ drag force remains almost as high in the

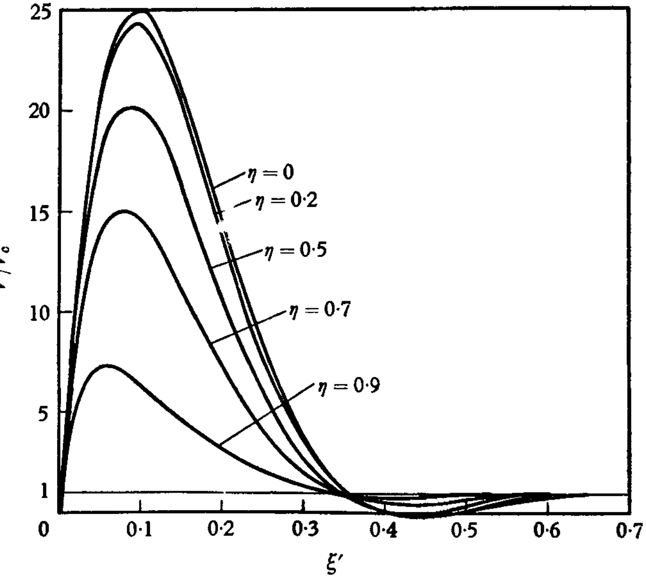

FIGURE 2. Case I: $d_A = 0$, $M = 100$. Graph of V/V_c against ξ' in the boundary layer on $\xi = -l$ for various values of η.

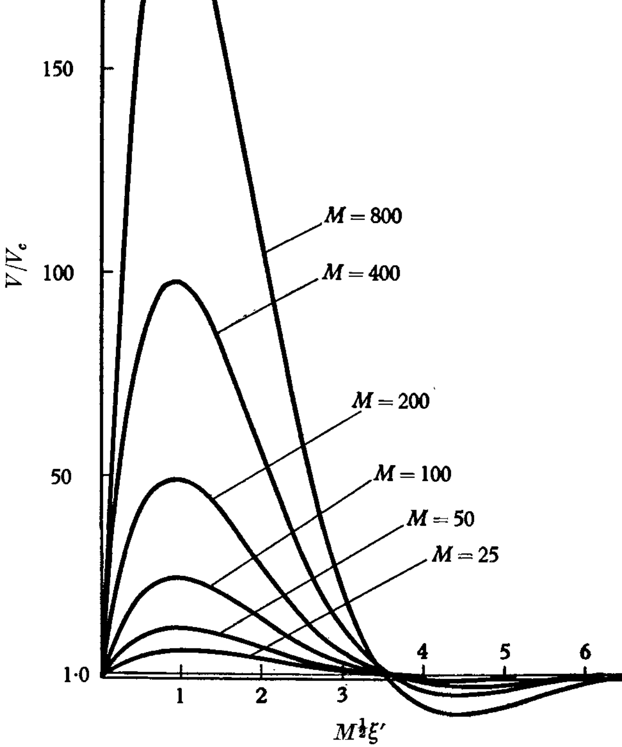

FIGURE 3. I: $d_A = 0$. Graph of V/V_c against $\sqrt{M}\xi'$ at η the boundary layer on $\xi = -l$ for various values of M

boundary layer as it is in the core and in the second case the $\mathbf{j} \times \mathbf{B}$ drag force decreases to zero at the walls, which explains why the velocities in the boundary layer are much less when $d_A = \infty$ than when $d_A = 0$. The reason for the large positive and negative velocities when $d_A = 0$ is difficult to explain simply, but it appears that relative to its value in the core, the $\mathbf{j} \times \mathbf{B}$ force *increases* at the outer edge of the boundary layer, where the negative velocity occurs, before it *decreases* near the wall, where the large positive velocities occur.

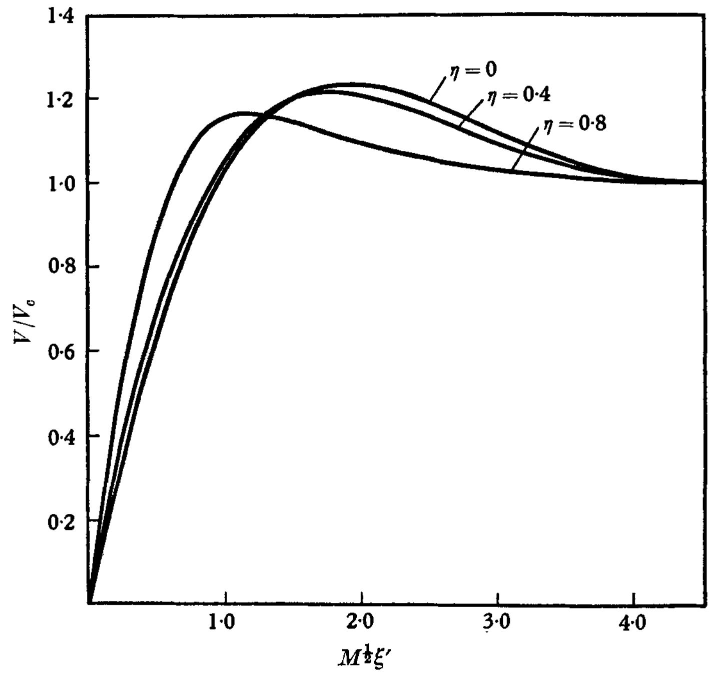

FIGURE 4. Case I: $d_A = \infty$. Graph of V/V_c against $\sqrt{M}\xi'$ in the boundary layer at $\xi = -l$ for various values of η at any value of M, provided $M \gg 1$.

Figures 2 and 4 show how little variation in velocity there is in the η-direction as compared with the ξ' direction which is to be expected since the magnetic field tends to damp only the vorticity perpendicular to it.

Williams has worked out an asymptotic expansion for Q when $d_A = \infty$ as $M \to \infty$ in terms of $1/M$. It is possible to use a simpler method than he used to derive the same result and this same method may also be used for any value of d_A.

We consider the expression for Q in equation (15) as $M \to \infty$ and make the same approximations as in equations (16) to (21). As $M \to \infty$, $M\gamma_j/\alpha_j \simeq M\lambda_j/\alpha_j$ and $\beta_j \simeq \lambda_j$, where $\lambda_j \simeq (\frac{1}{2}\alpha_j M)^{\frac{1}{2}}$. Hence for low values of j, such that $\alpha_j \gg M$, $M\gamma_j/\alpha_j \gg \beta_j$. Also, as $M \to \infty$, $\cosh 2\beta_j l \simeq \sinh 2\beta_j l \simeq \frac{1}{2}\exp(2l\lambda_j)$ and hence $\cosh 2\beta_j l \gg \cos 2\beta_j l$ and $\sinh 2\beta_j l \gg \sin 2\gamma_j l$. Therefore from equation (15), as $M \to \infty$

$$Q \sim \sum_{j=0}^{\infty} \frac{8}{\alpha_j^2(\alpha_j^2+M^2)} \left\{ l + \frac{(M\lambda_j/\alpha_j)\,\frac{1}{2}\{\exp(2l\lambda_j)\}}{2\lambda_j^2\,\frac{1}{2}[\{\exp(2l\lambda_j)\}(1+d_A\lambda_j)]} - \frac{\alpha_j d_A\,\frac{1}{2}\{\exp(2l\lambda_j)\}}{(\alpha_j^2+M^2)^{\frac{1}{2}}\,\frac{1}{2}[\{\exp(2l\lambda_j)\}(1+d_A\lambda_j)]} \right\},$$

and hence,

$$Q \sim \sum_{j=0}^{\infty} \frac{8}{\alpha_j^2(\alpha_j^2+M^2)}\left\{l+\frac{M^{\frac{3}{2}}(2\alpha_j)^{-\frac{1}{2}}-d_A\alpha_j^2}{M\alpha_j\{1+d_A(\frac{1}{2}\alpha_j M)^{\frac{1}{2}}\}}\right\}. \tag{22}$$

The first term in this expression represents the velocity flux due to Hartmann flow between the planes $\eta = \pm 1$, while the second term is the change due to the boundary layers on the walls AA. Note that the form of the second term depends

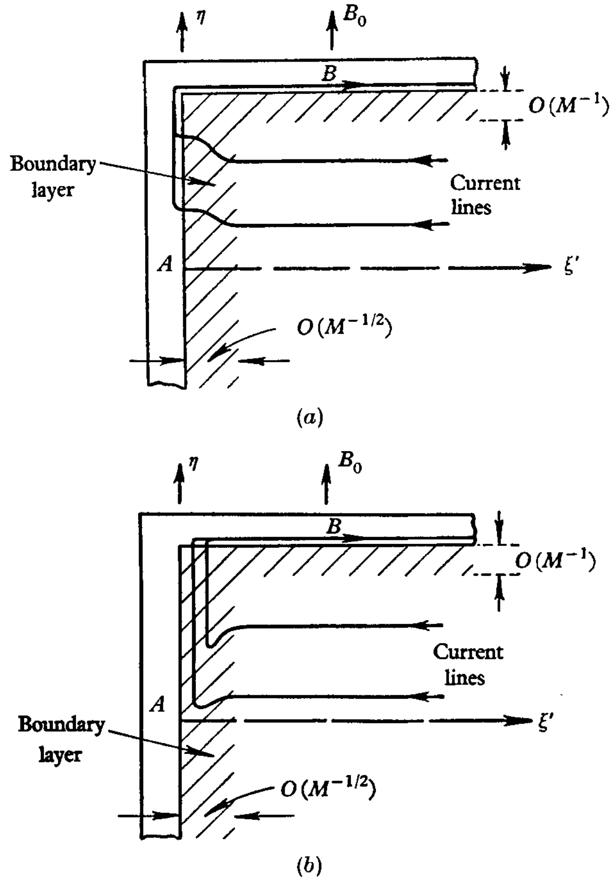

FIGURE 5(*a*). Cross-section of the duct when $d_A = \infty$ and $d_B = \infty$ ($M \gg 1$). (Not to scale.) (*b*) Cross-section of the duct when $d_A = 0$ and $d_B = \infty$ ($M \gg 1$). (Not to scale.)

on the value of $d_A/M^{\frac{1}{2}}$, the ratio of the conductance of the wall to that of the boundary layer. If $d_A = \infty$,

$$Q \sim \frac{4l}{M^2}\left(1-\frac{1}{M}\right)-\sum_{j=0}^{\infty}\frac{8\sqrt{2}}{M^{\frac{7}{2}}\alpha_j^{\frac{3}{2}}} \simeq \frac{4l}{M^2}\left(1-\frac{1}{M}\right)-\frac{32}{M^{\frac{7}{2}}\pi^{\frac{3}{2}}}\sum_{j=0}^{\infty}\frac{1}{(2_j+1)^{\frac{3}{2}}},$$

and summing the series leads to

$$Q \sim \frac{4l}{M^2}\left(1-\frac{1}{M}-\frac{2{\cdot}40}{lM^{\frac{3}{2}}}+O\left(\frac{1}{M^2}\right)\right).$$

Hence the mean velocity

$$\bar{v}_z \sim \frac{(-\partial p/\partial z)\,a^2}{\eta M^2}\left(1-\frac{1}{M}-\frac{2{\cdot}40a}{bM^{\frac{3}{2}}}\right). \tag{23}$$

In the exact expression derived by Williams the coefficient of the third term was 2·43.

If $d = 0$,

$$Q \sim \frac{4l}{M^2}\left(1-\frac{1}{M}\right)+\sum_{j=0}^{\infty}\frac{64}{M^{\frac{3}{2}}\,(2_j+1)^{\frac{7}{2}}}$$
$$\sim \frac{1{\cdot}20}{M^{\frac{3}{2}}}+\frac{4l}{M^2}+O\left(\frac{1}{M^{\frac{5}{2}}}\right).$$

Hence the mean velocity

$$\bar{v}_z \sim \frac{(-\partial p/\partial z)\,a^2}{\eta}\left\{\frac{a}{b}\frac{0{\cdot}30}{M^{\frac{3}{2}}}+\frac{1}{M^2}+\frac{a}{b}O\left(\frac{1}{M^{\frac{5}{2}}}\right)\right\}. \tag{24}$$

These results further demonstrate the interesting physical effects due to the conducting walls. From (23) we see that the velocity deficiency in the boundary layers on AA, i.e.

$$\int_{-1}^{+1}\int_{-l}^{+l}(V_c - V)\,d\eta\,d\xi,$$

is $O(M^{-\frac{7}{2}})$, when all the walls are perfectly conducting, i.e. $d_A = \infty$. This is less than that in the case of the rectangular duct with non-conducting walls where it is $O(M^{-\frac{3}{2}})$ (Shercliff 1953). The reduced velocity deficiency is due to the velocity in the boundary layers showing an overshoot; relative to its value in the core the velocity first decreases, then increases above its value in the core, and finally decreases to zero at the wall. In fact there are an infinite number of fluctuations in the velocity profile between the overshoot, just referred to, and the core, but they are sufficiently small for us to ignore them (see figure 4). Note that in both these cases the thickness of the boundary layers is $O(M^{-\frac{1}{2}})$.

When $d_A = 0$ the velocity 'deficiency', as defined above, is negative and we find that for a reasonably square duct ($a/b \gg M^{-\frac{1}{2}}$) most of the flow (equation (24)) is *in the boundary layers* on AA. If the boundary layer has thickness $O(M^{-\frac{1}{2}})$ and the velocity in the boundary layer is $O(MV_c) = O(M^{-1})$ then the velocity flux through the boundary layer is $O(M^{-\frac{3}{2}})$ as compared with $O(M^{-2})$ for the total velocity flux in the core. A practical consequence of this would be that, whatever the value of d_A, for a given pressure gradient a system of thin insulating baffles placed parallel to AA at a distance $O(aM^{-\frac{1}{2}})$ apart would promote a greater volume flow rate by creating more boundary layers. The explanation is that the dominant retarding force on the core flow is electro-magnetic rather than viscous and the baffles will reduce the currents and hence also the electromagnetic retarding force. Provided the baffles are at least $O(aM^{-\frac{1}{2}})$ apart, the decrease in electromagnetic drag will be greater than the increase in viscous drag.

4. Case II

In this case walls AA are non-conducting, $d_A = 0$, and walls BB have arbitrary conductivity, d_B. The boundary conditions on V and H are

$$\left.\begin{aligned} &\text{at } \eta = \pm 1, \quad V = 0, \quad \partial H/\partial \eta = \mp H/d_B,\\ \text{and}\quad &\text{at } \xi = \pm l, \quad V = 0, \quad H = 0. \end{aligned}\right\} \qquad (25$$

We satisfy the boundary conditions on $\xi = \pm l$ by expressing V and H as Fourier series in ξ, with coefficients functions of η,

$$V = \sum_{k=0}^{\infty} v_k(\eta)\cos\alpha_k\xi, \quad H = \sum_{k=0}^{\infty} h_k(\eta)\cos\alpha_k\xi, \qquad 1 = \sum_{k=0}^{\infty} a_k\cos\alpha_k\xi,$$

where
$$\alpha_k = (k+\tfrac{1}{2})\frac{\pi}{l} \quad \text{and} \quad a_k = \frac{2(-1)^k}{\alpha_k l}.$$

Substituting these values for V and H into (7) and (8) again leads to two ordinary differential equations for v_k and h_k

$$v_k'' - \alpha_k^2 v_k + Mh_k' = -a_k,$$
$$b_k'' - \alpha_k^2 b_k + Mv_k' = 0.$$

The solutions of these equations which satisfy the boundary conditions (25) involve some formidable algebra. The results are

$$V = \sum_{k=0}^{\infty}\frac{2(-1)^k\cos\alpha_k\xi}{l\alpha_k^3}\left[1 - \frac{(1+\tanh r_{2k}/d_B r_{2k})\cosh r_{1k}\eta}{(\cosh r_{1k})(M^2+4\alpha_k^2)^{\frac{1}{2}}/r_{2k} + \sinh(r_{1k}+r_{2k})/d_B r_{2k}\cosh r_{2k}} - \frac{(1+\tanh r_{1k}/d_B r_{1k})\cosh r_{2k}\eta}{(\cosh r_{2k})(M^2+4\alpha_k^2)^{\frac{1}{2}}/r_{1k} + \sinh(r_{1k}+r_{2k})/d_B r_{1k}\cosh r_{1k}}\right], \tag{26}$$

$$H = \sum_{k=0}^{\infty}\frac{2(-1)^k\cos\alpha_k\xi}{l\alpha_k^3}\left[\frac{(1+\tanh r_{2k}/d_B r_{2k})\sinh r_{1k}\eta}{(\cosh r_{1k})(M^2+4\alpha_k^2)^{\frac{1}{2}}/r_{2k} + \sinh(r_{1k}+r_{2k})/d_B r_{2k}\cosh r_{1k}} - \frac{(1+\tanh r_{1k}/d_B r_{1k})\sinh r_{2k}\eta}{(\cosh r_{2k})(M^2+4\alpha_k^2)^{\frac{1}{2}}/r_{1k} + \sinh(r_{1k}+r_{2k})/d_B r_{1k}\cosh r_{1k}}\right], \tag{27}$$

where $r_{1k}, r_{2k} = \frac{1}{2}(\pm M + \{M^2+4\alpha_k^2\}^{\frac{1}{2}})$.

When $d_B = 0$ then all the walls are non-conducting which is the case analyzed by Shercliff (1953). Putting $d_B = 0$ in the above formulae and adding V to H gives Shercliff's result (equation (15) in his paper).

When $d_B = \infty$ the duct is the same as that analyzed in §3 when $d_A = 0$. It would be desirable to check that the two solutions were the same but this proves difficult, since to examine the above solutions near $\xi = \pm l$ the higher harmonics in the expansions of V and H have to be considered. It would be easier to check that the expressions for Q derived from the two solutions are the same. Using (26), when $d_B = \infty$, gives

$$Q = \int_{-1}^{+1}\int_{-l}^{+l} V\,d\eta\,d\xi = \sum_{k=0}^{\infty}\frac{8}{l\alpha_k^4}[1 + r_{2k}\tanh r_{1k}/r_{1k}(M^2+4\alpha_k^2)^{\frac{1}{2}} - r_{1k}\tanh r_{2k}/r_{2k}(M^2+4\alpha_k^2)^{\frac{1}{2}}]. \tag{28}$$

We may expand this expression in terms of $1/M$ as $M \to \infty$, using the methods of Williams (1963) and it is easily seen that the leading term is $O(M^{-\frac{3}{2}})$ in agreement with the previous result (24). The significance of this term has already been discussed in §3.

From (26) we may see that V and Q depend on $d_B M$, the ratio of the conductance of the walls BB to that of the boundary layers on BB. Decreasing $d_B M$ makes the current induced in the core return through the Hartmann layers on BB and hence reduces the electro-magnetic drag on the flow. This in turn damps out the sinusoidal form of the boundary layers on AA and for some finite d_B, no negative velocities will be induced in these layers.

In examining the case when the walls are non-conducting, Shercliff first derived the exact solution for $(V+B)$, but found that this solution gave little information about the boundary layers at $\xi = \pm l$ owing to the slow convergence of the series. Then by assuming that, in the boundary layers on $\xi = \pm l$,

$$\frac{\partial^2(V+B)}{\partial\eta^2} \ll \frac{\partial^2(V+B)}{\partial\xi^2},$$

he found a self-similar solution for $(V+B)$. Thence he was able to work out the velocity deficiency in the boundary layer. Shercliff's method is not applicable to cases other than $d_B = 0$ and $d_A = 0$; no other type of self-similar solution has yet been found.

5. Conclusion

Though this study is far from complete, it does indicate the need for further theoretical and experimental study of MHD flows in ducts with conducting walls. First, it is not difficult in a mercury experiment to raise M to values greater than 100, and the effects predicted by the theory should be observable.

Secondly, the stability of the boundary layers on the walls AA in the presence of excess velocity and reversed flow needs theoretical examination. The analysis of the steady-state duct flow problem does not depend on the value of the Reynolds number R or the magnetic Reynolds number R_m, but the stability of such a flow depends on R, R_m, and M. In most practical situations $R_m \ll 1$ and we can ignore the Alfvén wave motions associated with $R_m \gg 1$. When $R_m \gg 1$ the stability analysis depends only on R and M. Lock (1955) has analyzed the stability of Hartmann flow and found that in realistic cases the magnetic field stabilizes the flow by its effect on the equilibrium velocity profile and not by inhibiting the growth of small disturbances, since this is dominated by viscous effects. We can then make some qualitative predictions about the stability of the flows studied above, based on our knowledge of the stability of boundary layers when there is no magnetic field.

Let us examine the stability of the boundary layers on the walls AA in a duct with perfectly conducting walls perpendicular to the field and insulating walls parallel to the field ($d_A = 0, d_B = \infty$). As $M \to \infty$, there is an increasing number of points of inflexion in the velocity profile which indicates that the higher M the lower the Reynolds number at which the flow in the boundary layers becomes unstable (figure 3). The degree to which the magnetic field is likely to be destabilizing depends also on the shape of the duct. If $a/b \ll M^{-\frac{1}{2}}$, a very thin duct with walls AA much shorter than walls BB, the mean velocity in the duct closely approaches the core velocity and most of the flow is in the core. (For $a/b \ll M^{-\frac{1}{2}}$ most of the flow is in the boundary layers on AA (§3)). Then the mean velocity in the boundary layers on the walls AA is $O(M)\,\bar{v}_z$, and since the thickness of these boundary layers is $O(aM^{-\frac{1}{2}})$, the Reynolds number of the boundary layer

$$R_{\text{b.l.}} = O(aM^{\frac{1}{2}})\,\bar{v}_z/\nu,$$

where ν is the kinematic viscosity. Hence

$$R_{\text{b.l.}} \simeq O(M^{\frac{1}{2}})\,R, \tag{29}$$

where R is the overall Reynolds number of the flow in the duct ($R = \bar{v}_z a/\nu$). Thus, for given R, $R_{\text{b.l.}}$ increases with M; hence the critical overall Reynolds number at which the boundary layer becomes unstable is reduced by increasing M. Note, however, that away from the remote walls AA the flow would be very stable.

Now consider an approximately square duct with $a/b = O(1)$. We see from equation (24) that in this case most of the flow is in the boundary layers on AA. The mean velocity in the boundary layers on AA is $O(M)\,v_c$, where v_c is the core velocity, and since the thickness of these boundary layers is $O(aM^{-\frac{1}{2}})$, the overall mean velocity is given by

$$v_z \simeq O[(Mv_c \times a^2M^{-\frac{1}{2}} + v_c \times ab)/ab] \simeq O[M^{\frac{1}{2}}\,v_c a/b].$$

Hence if $a/b = O[1]$, $R \simeq O[M^{\frac{1}{2}}av_c/\nu]$ and since $R_{\text{b.l.}} \simeq O[M^{\frac{1}{2}}av_c/\nu]$,

$$R \simeq R_{\text{b.l.}}. \tag{30}$$

Thus for this type of duct for given R, $R_{\text{b.l.}}$ does not increase with M. Comparing (29) and (30) indicates that the thinner the duct the more the magnetic field tends to destabilize the flows in the boundary layers on AA.

It is important to realize that the forms of the velocity profiles are functions of M and not R. Thus velocity overshoot and reversed flow can occur in the boundary layers on AA at arbitrarily small Reynolds number. We cannot assume, therefore, that these boundary layers are always unstable as $M \to \infty$: they will probably be stable at sufficiently small Reynolds numbers, whatever the value of M.

When the walls of the duct are all perfectly conducting ($d_A = d_B = \infty$) the velocity profile of the boundary layers also contains points of inflexion (figure 4) and hence raising M reduces the Reynolds number at which these boundary layers become unstable. But in this case for $M \gg 1$ the velocity in the boundary layers on AA is of the same order as the core velocity and since the boundary layer thickness is $O(aM^{-\frac{1}{2}})$,

$$R_{\text{b.l.}} = O(M^{-\frac{1}{2}})\,R.$$

(Provided $a/b < M^{\frac{1}{2}}$ the shape of the duct does not matter.) Therefore in contrast to the former case ($d_A = 0$; $d_B = \infty$), raising M at given R may first tend to *destabilize* the flow in the boundary layers on AA and then to *stabilize* it.

The only tentative conclusion we can draw from this qualitative analysis is that, for flow in a rectangular duct with conducting walls, the value of the overall Reynolds number at which the boundary layers on walls AA become unstable decreases as the Hartmann number increases. This may be contrasted to the case of flow in a plane channel where it has been shown, both theoretically and experimentally, that the magnetic field *stabilizes* the flow.

Thirdly, we have only discussed flows in ducts whose walls are either perfect conductors of insulators and it would be of interest to study the cases where the walls have finite conductivity. When all the walls are non-conducting and $M \gg 1$, the velocity profile in the boundary layers on the walls AA has no points of inflexion and the flow in such a duct is probably stabilized by the magnetic field. Hence it is likely that uniformly lowering the conductivity of the walls will tend to stabilize the flows in the boundary layers. And lastly we have not considered contact resistance, though it would not be difficult to include it in the analysis.

I should like to express by thanks to Dr J. A. Shercliff for the help he has given me and for the interest he has shown in this work.

The work has been carried out under the sponsorship of the Central Electricity Research Laboratories, and is published by permission of the Central Electricity Generating Board.

REFERENCES

Braginskii, S. I. 1960 *Sov. Phys. J.E.T.P.* **10**, 1005.

Chang, C. C. & Lundgren, T. S. 1961 *Z. angew. Math. Phys.* **12**, 100.

Fabri, J. & Siestrunck, R. 1960 *Bull. Assoc. Tech. Marit. Aero.* no. 60, 333.

Gold, R. A. 1962 *J. Fluid Mech.* **13**, 505.

Grinberg, G. A. 1961 *Appl. Math. Mech. (Prik. Mat. Mek.)*, **25**, 1536.

Grinberg, G. A. 1962 *Appl. Math. Mech. (Prik. Mat. Mek.)*, **26**, 106.

Lock, R. C. 1955 *Proc. Roy. Soc.* A, **233**, 105.

Sakao, F. 1962 *J. Aero Space Sci.* **29**, 246.

Shercliff, J. A. 1953 *Proc. Camb. Phil. Soc.* **49**, 136.

Shercliff, J. A. 1956 *J. Fluid Mech.* **1**, 644.

Shercliff, J. A. 1962*a* *J. Fluid Mech.* **13**, 513.

Shercliff, J. A. 1962*b* *The Theory of Electromagnetic Flow Measurement*. Cambridge University Press.

Uflyand, Y. S. 1961 *Sov. Phys.-Tech. Phys.* **5**, 1194.

Williams, W. E. 1963 *J. Fluid Mech.* **16**, 262.